SAT Solving

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Outline

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The SAT Problem

- A set of boolean variables: $X = \{x_1, x_2, ..., x_n\}$.
- Literals: $x_1, \neg x_1, x_2, \neg x_2, ..., x_n, \neg x_n$
- A set of Clauses: $x_1 \vee \neg x_2, x_2 \vee x_3, x_2 \vee \neg x_4, \neg x_1 \vee \neg x_3 \vee x_4, ...$
- A Conjunctive Normal Form (CNF) formula:

$$\varphi = (x_1 \vee \neg x_2) \wedge (x_2 \vee x_3) \wedge (x_2 \vee \neg x_4) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$$

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- The satisfiability problem (SAT): test whether there exists an assignment of truth values to the variables in F under which F evaluates to true.

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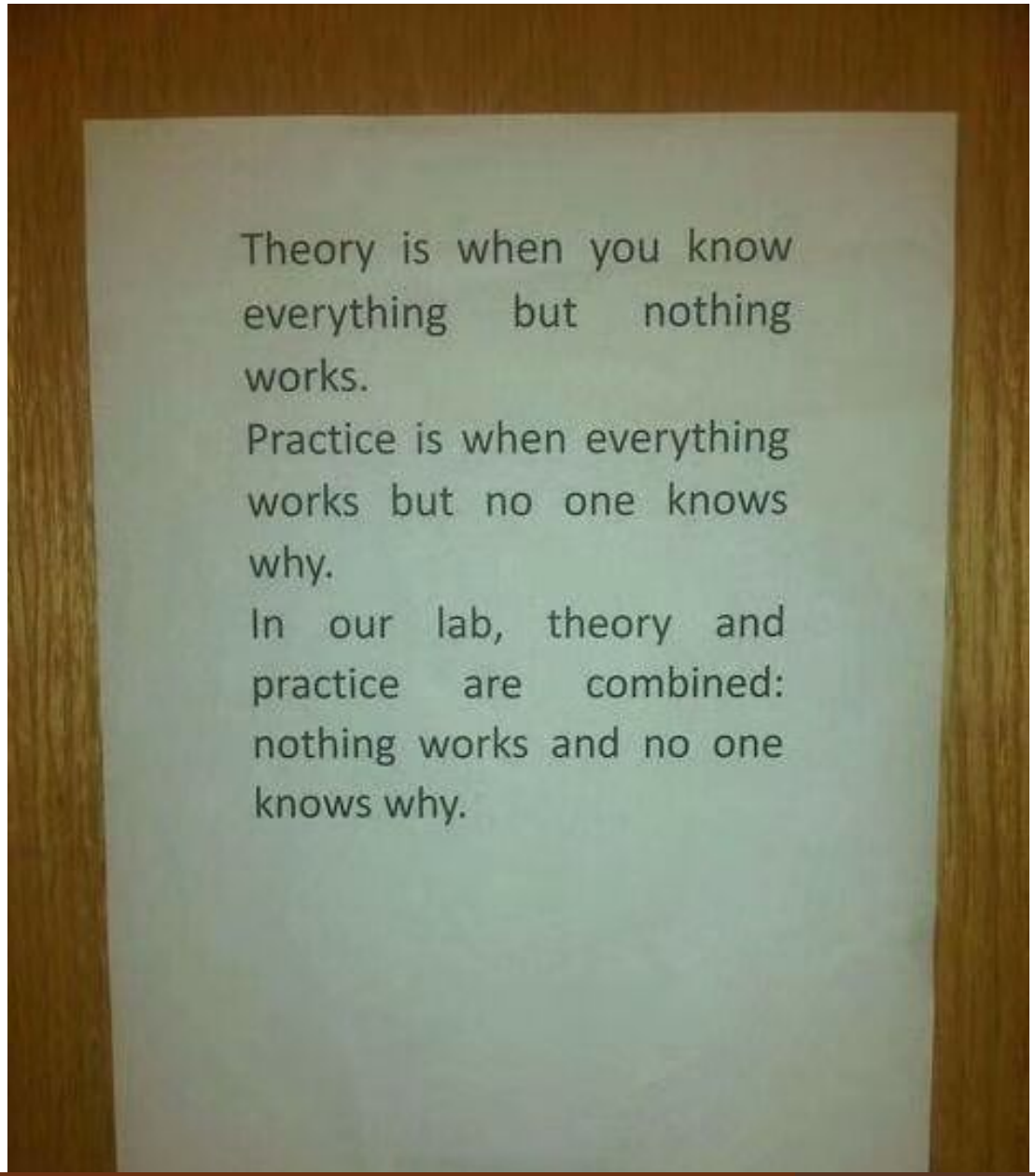
- Many important applications
- Competitions, open source benchmarks and solvers

About SAT: theory vs. practice



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- The only hope for a practical solver is that by being smart in the search.
- Modern SAT solvers can solve formulas with up to millions of variables.

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- Radio spectrum reallocation of USA (7-billion dollars earnings) by a software named SATFC [Leyton-Brown et al., PNAS 2017], in which a core SAT solver is our solver DCCASat.

Solving SAT

Methods for solving SAT can be classified to two classes:

- Complete methods: DPLL $\longrightarrow$ Conflict Driven Clause Learning (CDCL)
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- Complete methods: DPLL $\longrightarrow$ Conflict Driven Clause Learning (CDCL)
  - Guarantee the correct solution when the algorithm terminates, but it can cost unreasonable long time
- Incomplete methods: mainly Local search
  - Can not prove the unsatisfiability of a formula, but may find a solution fast for certain types of instances

Backtracking framework of Complete Search for SAT

- search space of a formula

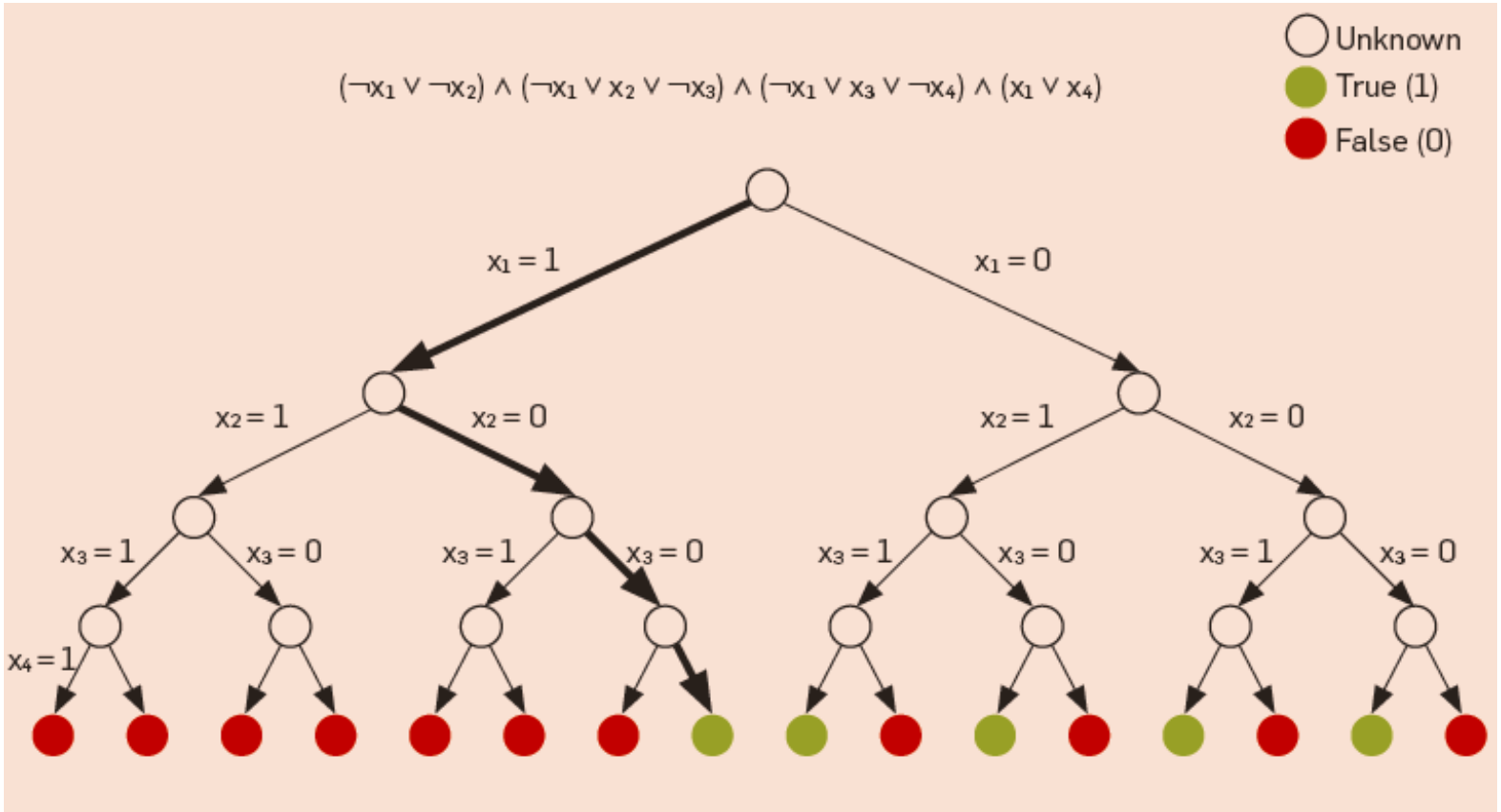

figure taken from [Sharad Malik, Lintao Zhang, CACM 2009]

Basic Rules

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  - A literal is pure if only occurs as a positive literal or as a negative literal in a CNF formula.
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  - A reference technique until the mid 90s; nowadays seldom used.

Basic Rules

- Unit Propagation
  - Given a unit clause, its only unassigned literal must be assigned value 1 for the clause to be satisfied.
  - E.g., for unit clause $(x_1 \vee \neg x_2 \vee \neg x_3)$, and $x_1 = 0$, $x_2 = 1$, then x_3 must be assigned value 0.

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  - Very important technique even nowadays.

DPLL Algorithm

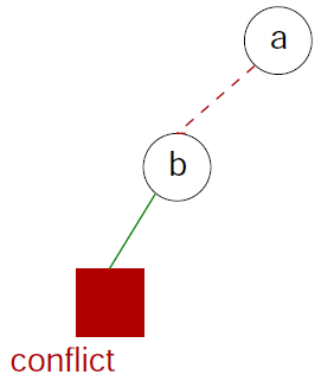
DPLL algorithm [Davis and Putnam 1960, Davis, Logemann and Loveland 1962]

- backtracking framework
- equipped with simple reasoning techniques including pure literals and unit propagations.
- heuristics for variable ordering and value ordering.

DPLL Algorithm

Example:

$$\begin{aligned}\varphi \quad = \quad & (a \vee \neg b \vee d) \wedge (a \vee \neg b \vee e) \wedge \\ & (\neg b \vee \neg d \vee \neg e) \wedge \\ & (a \vee b \vee c \vee d) \wedge (a \vee b \vee c \vee \neg d) \wedge \\ & (a \vee b \vee \neg c \vee e) \wedge (a \vee b \vee \neg c \vee \neg e)\end{aligned}$$



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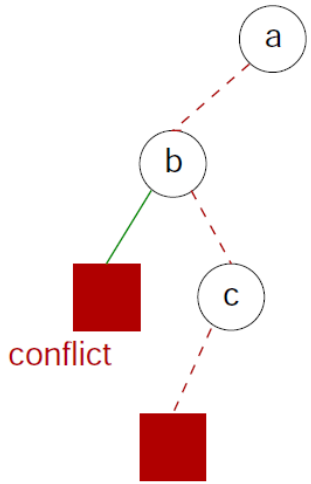

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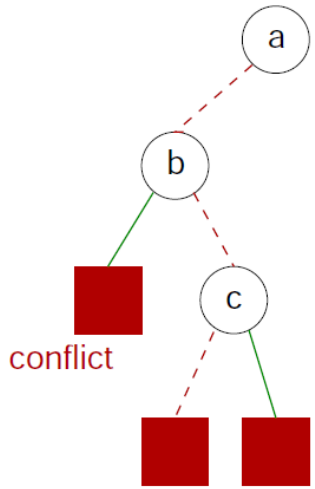

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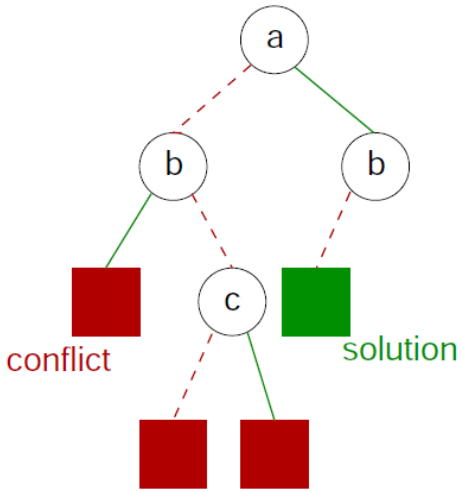

From DPLL to CDCL

Conflict Driven Clause Learning (CDCL)

- Learning and non-chronological backtracking, 1996
- Conflict Driven Clause Learning (CDCL), 1999
- Efficient data structures
- Restart

Clause Learning

Clause Learning:

- During backtrack search, for each conflict learn new clause, which explains and prevents repetition of the same conflict

$$\varphi = (\ a\ \vee b) \wedge (\neg b \vee\ \ c\ \vee d) \wedge (\neg b \vee e) \wedge (\neg d \vee \neg e \vee\ \ f\ ) \ldots$$

  - Assume decisions $c = 0$ and $f = 0$
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  - Assume decisions $c = 0$ and $f = 0$
  - Assign $a = 0$ and imply assignments
  - A conflict is reached: $(\neg d \vee \neg e \vee f)$ is unsatisfied
  - $(a = 0) \wedge (c = 0) \wedge (f = 0) \Rightarrow (\varphi = 0)$
  - $(\varphi = 1) \Rightarrow (a = 1) \vee (c = 1) \vee (f = 1)$

  - Learn new clause $(a \vee c \vee f)$

Local Search for SAT

Local search for SAT: [Selman et al. 1992; Gu 1992]

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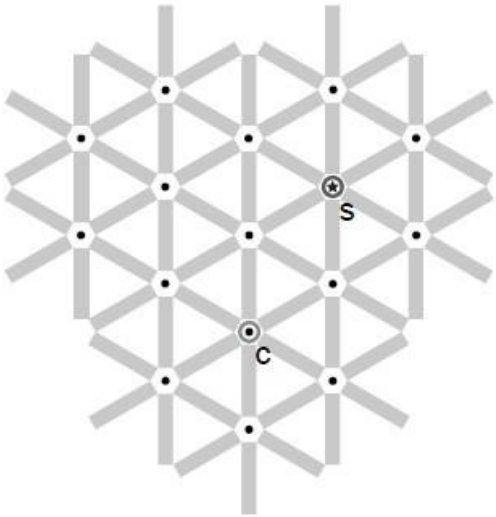
Local search is incomplete

- It cannot prove the unsatisfiability of a formula
- It cannot guarantee to find a solution even for satisfiable formula, but adding random walks can make it Probabilistic Approximate Complete.

Local Search for SAT



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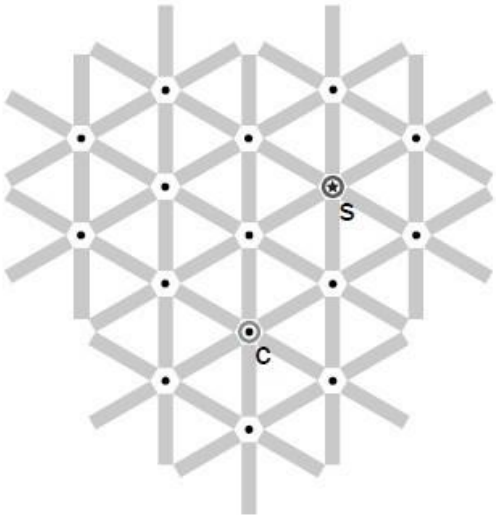

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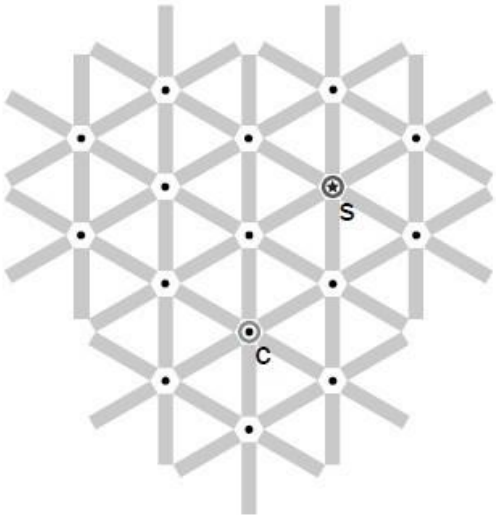

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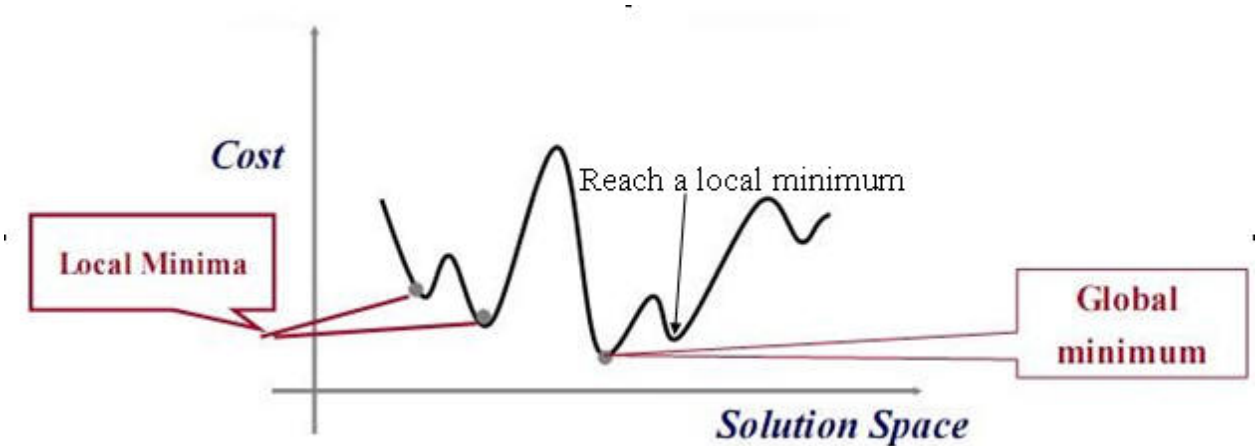

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- $break(x)$: the number of currently satisfied clauses that would become unsatisfied by flipping x.
- $score(x)$: $cost(F, \alpha) - cost(F, \alpha')$, i.e., the number of currently unsatisfied clauses minus the number of unsatisfied clauses if x were to be flipped.

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- $score(x)$: $cost(F, \alpha) - cost(F, \alpha')$, i.e., the number of currently unsatisfied clauses minus the number of unsatisfied clauses if x were to be flipped.
- It is easy to see that $score(x) = make(x) - break(x)$.

Scoring Functions of Variables

A scoring function can be a simple variable property or any mathematical expression with one or more properties.

- $break(x)$
- $score(x)$
- $age(x)$
- $score(x) + age(x)/T$
- $score(x)^a$
- $b^{score(x)}$
- ...

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A key factor degrading the performance of local search

- wastes time on revisiting
- prevents it from getting out of local minima

Previous Methods

- Naive methods
  - Random walk
  - allow non-improving steps with a probability (simulated annealing)
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- The tabu mechanism forbids reversing the recent changes, where the strength of forbidding is controlled by a parameter called tabu tenure [Fred Glover, 1989].

Configuration Checking

Address cycling problem by Configuration Checking (CC) [Artif. Intel. 2011].

CC is found effective for the following types of problems:

- Assignment Problems: to find an assignment to all variables such that satisfies the constraints (and optimized).
- Subset Problems: to find a subset from a universe set such that satisfies the constraints (and optimized).

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CC relies on the concept of the "configuration" of variables, which is some form of circumstance of the variables.

Definition

The *configuration* of a variable x is a vector C_x consisting of truth value of all x's neighboring variables.

Intuition of CC

Configuration Checking for SAT: For a variable x, if the configuration of x has not changed since x's last flip, then x is forbidden to be flipped.

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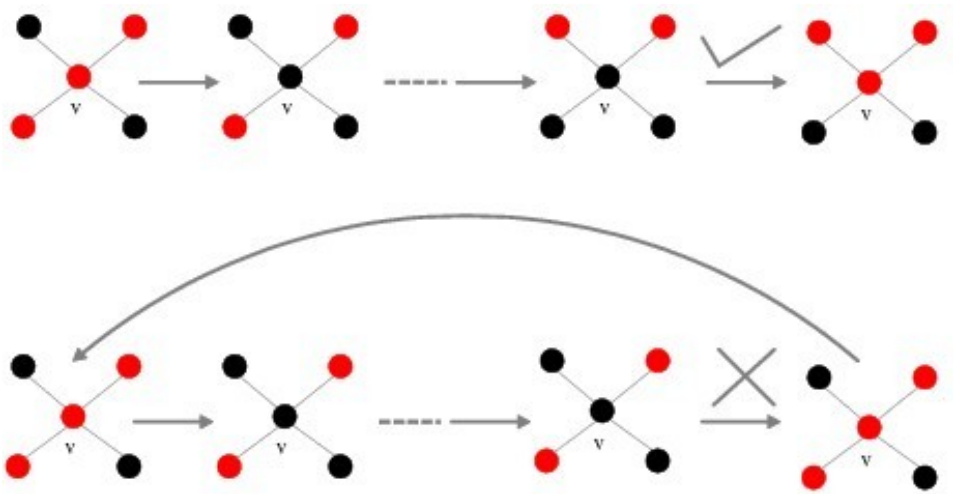

reduce local structure cycles

Variants of CC for SAT

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Definition

In Clause States based CC (CSCC), the configuration of a variable x is a vector that consists of the states of all the clauses in which x appears.

Combination of CC strategies

We can also combine different CC strategies in one algorithm.

Double Configuration Checking (DCC) for SAT:

Lemma

If a variable is configuration changed w.r.t. CSCC criterion, then it is is configuration changed w.r.t. NVCC, while the reverse is not true.

DCC heuristic: first selects a CSCC variable; if no such variable, selects a NVCC variable.

Success of CC for SAT

- Many participant solvers use CC in SAT Competitions from 2012, including 10 medal-awarded ones.
- AAAI 2013 Tutorial Forum: "It is outstanding. It is likely changing the game."
- Used in solving real world application projects.

CC and variable neighborhood

The effectiveness of CC is related to the neighborhood of variables.

Theorem

For a random k-SAT formula $F_k(n, m)$, if $ln(n-1) < \frac{k(k-1)r}{(n-1)}$, then CC degrades forbids only one variable each step.

When CC Becomes Ineffective

$f(n) = ln(n-1) - \frac{k(k-1)r}{n-1}$ is a monotonic increasing with n $(n > 1)$.
$f(n) < 0$ iff $n \leq \lfloor n^* \rfloor$, where n^* is a real number s.t. $f(n^*) = 0$.

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$f(n) < 0$ iff $n \leq \lfloor n^* \rfloor$, where n^* is a real number s.t. $f(n^*) = 0$.
For phase-transition area of k-SAT, such n^* values are as follows.

Formulas	3-SAT ($r = 4.2$)	4-SAT ($r = 9.0$)	5-SAT ($r = 20$)	6-SAT ($r = 40$)	7-SAT ($r = 85$)
n^*	11.652	32.348	90.093	223.095	564.595

Table: The n^* value such that when $n < n^*$, the CC strategy degrades to a trivial case.

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- Automatic algorithm configuration.

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Thank you!