One hundred prisoners and a lightbulb

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A group of 100 prisoners, all together in the prison dining area, are told that they will be all put in isolation cells and then will be interrogated one by one in a room containing a light with an on/off switch. The prisoners may communicate with one another by toggling the light-switch (and that is the only way in which they can communicate). The light is initially switched off. There is no fixed order of interrogation, or interval between interrogations, and the same prisoner may be interrogated again at any stage. When interrogated, a prisoner can either do nothing, or toggle the light-switch, or announce that all prisoners have been interrogated. If that announcement is true, the prisoners will (all) be set free, but if it is false, they will all be executed. While still in the dining room, and before the prisoners go to their isolation cells (forever), can the prisoners agree on a protocol that will set them free?

100 prisoners — not a solution

Let there be one prisoner:

Protocol: *If a prisoner enters the interrogation room, he announces that all prisoners have been interrogated.*

Let there be two prisoners:

Protocol: *If a prisoner enters the interrogation room and the light is off, he turns it on, if a prisoner enters the interrogation room and the light is on and he has not turned it on, he announces that all prisoners have been interrogated.*

Let there be three prisoners:

Protocol: ...

100 prisoners — not a solution

It does not help to ...

- feel if the lightbulb is warm
- smash the lightbulb (works for 3, not for more)
- keep track of the time
- watch from your isolation cell if the light is on or off ...
  You cannot see a thing! It is an isolation cell!
- watch from your isolation cell if a prisoner is accompanied to the interrogation room ...
  You cannot see a thing! It is an isolation cell!
- hear from your isolation cell the click of the lightswitch ...
  You cannot hear a thing! It is an isolation cell!

Protocol for $n \geq 3$ prisoners

The n prisoners appoint one amongst them as the counter. The non-counting prisoners are the followers. The followers follow the following protocol: the first time they enter the room when the light is off, they turn it on; on all other occasions, they do nothing. The counter follows a different protocol. When the light is on when he enters the interrogation room, he turns it off. When he turns off the light for the $(n-1)$st time, he announces that everybody has been interrogated.

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Let us picture a number of executions of this protocol for $n = 3$. The upper index: state of the light. The lower index: the number of times the light has been turned off. Anne is the counter.

— ${}^0\mathrm{Bob}^1\mathrm{Anne}_1^0\mathrm{Caro}^1\mathrm{Anne}_2^0$
— ${}^0\mathrm{Anne}^0\mathrm{Bob}^1\mathrm{Caro}^1\mathrm{Anne}_1^0\mathrm{Bob}^0\mathrm{Anne}_1^0\mathrm{Caro}^1\mathrm{Caro}^1\mathrm{Bob}^1\mathrm{Bob}^1\mathrm{Anne}_2^0$
— ${}^0\mathrm{Bob}^1\mathrm{Anne}_1^0\mathrm{Bob}^0\mathrm{Caro}^1\mathrm{Bob}^1\mathrm{Anne}_2^0$

If the scheduling is fair, then the protocol will terminate.

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What if it is not known whether the light is initially on?
Same count, you may get hanged (namely if light was on).
One higher, you may never terminate (namely if light was off).

— (light on, same) ${}^1\mathrm{Anne}^0_1\mathrm{Caro}^1\mathrm{Anne}^0_2$ ☹
— (light on, higher) ${}^1\mathrm{Anne}^0_1\mathrm{Caro}^1\mathrm{Anne}^0_2\mathrm{Bob}^1\mathrm{Anne}^0_3$ ☺
— (light off, same) ${}^0\mathrm{Bob}^1\mathrm{Anne}^0_1\mathrm{Caro}^1\mathrm{Anne}^0_2$ ☺
— (light off, higher) ${}^0\mathrm{Bob}^1\mathrm{Anne}^0_1\mathrm{Caro}^1\mathrm{Anne}^0_2\mathrm{Bob}^0\mathrm{Anne}^0_2\ldots$ ☹

100 prisoners — solution if light may be on or off

The n prisoners appoint one amongst them as the counter. The non-counting prisoners are the followers. The followers follow the following protocol: the ~~first time~~ first two times they enter the room when the light is off, they turn it on; on all other occasions, they do nothing. The counter follows a different protocol. When the light is on when he enters the interrogation room, he turns it off. When he turns off the light for the ~~$(n-1)$st time~~ $(2n-2)$nd time, he announces that everybody has been interrogated.

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For $n = 100$, the next entry after 198 switches:

— light was off and 99 non-counters have been interrogated twice
— light was on and 98 non-counters twice and one once only.

Either way is fine!

Followers can also count

A follower may know before the counter that everybody has been interrogated. — $^0\mathrm{Bob}^1\mathrm{Anne}^0_1\mathrm{Bob}^0\mathrm{Caro}^1\mathrm{Bob}^1\mathrm{Anne}^0_2$

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Keep track how often you observe the light being on after being off. As such also counts (i) if the light is on at your first interrogation and (ii) the (single) occasion that you turn on the light yourself. Call this the number of switches.

The protocol is now: counter and followers behave as in the standard protocol; and the followers also count the number of switches. A follower announces that all prisoners have been interrogated after $n-1$ switches.

— ${}^0\text{Bob}^1\text{Anne}^0_1\text{Caro}^1\textbf{Anne}^0_2$
— ${}^0\text{Anne}^0\text{Bob}^1\text{Caro}^1\text{Anne}^0_1\text{Bob}^0\text{Anne}^0_1\textbf{Caro}^1\text{Caro}^1\text{Bob}^1\text{Bob}^1\text{Anne}^0_2$
— ${}^0\text{Bob}^1\text{Anne}^0_1\text{Bob}^0\text{Caro}^1\textbf{Bob}^1\text{Anne}^0_2$

How likely is this to happen for 3 prisoners? For 100 prisoners?

A probabilistic protocol with uniform roles

Each prisoner holds a token initially worth one point. Turning the light on if it is off, means dropping one point. Leaving the light on if it is on, means not being able to drop one point. Turning the light off if it is on, means collecting one point. Leaving the light off if it is off, means not being able to collect one point.

Protocol: *Your* token *is the sum of the points you hold plus the state of the light* (1 *if on and* 0 *if off). Let your token be m. Let a function* $Pr : \{0, ..., n\} \to [0, 1]$ *be given, with* $Pr(0) = Pr(1) = 1$, $0 < Pr(x) < 1$ *for* $x \neq 0, 1, n$, *and* $Pr(n) = 0$. *Drop your point with probability* $Pr(m)$, *otherwise, collect it. The protocol terminates once a prisoner has collected n points.*

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$Pr(0) = Pr(1) = 1$, $Pr(2) = 0.5$, $Pr(3) = Pr(4) = 0$. *Lower index*: the number of points held. *Upper index*: state of the light. *Bold*: prisoner collects point. *Not bold*: drops point.

${}^{0}\mathrm{Anne}_{0}^{1}\mathrm{Bob}_{1}^{1}\mathbf{Caro}_{2}^{0}\mathrm{Dick}_{0}^{1}\mathbf{Bob}_{2}^{0}\mathbf{Caro}_{2}^{0}\mathrm{Caro}_{1}^{1}\mathbf{Bob}_{3}^{0}\mathrm{Caro}_{0}^{1}\mathbf{Bob}_{4}^{0}$

100 prisoners — synchronization

Assume a single interrogation per day takes place.
When can the prisoners expect to be set free from prison?

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non-counter / counter / another non-counter / counter / etc.

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Summation:

$$\sum_{i=1}^{99}\left(\frac{100}{i}+\frac{100}{1}\right) = 99\cdot 100+100\cdot\sum_{i=1}^{99}\frac{1}{i} = 9,900+518 \text{ days} \approx 28.5 \text{ years}$$

100 prisoners — improvements given synchronization

Dynamic counter assignment (protocol in two stages):

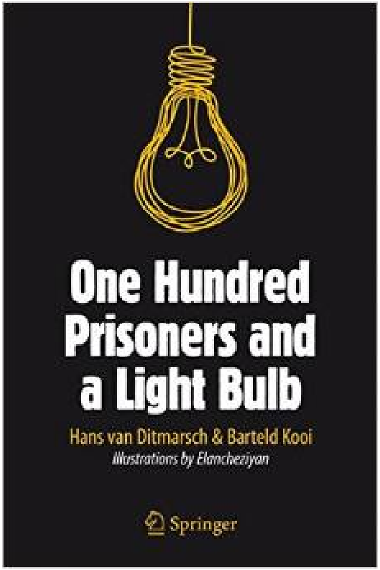
- stage 1, 99 days: the first prisoner to enter the room twice turns on the light. (Expectation: 13 days.)
- stage 1, day 100: if light off, done; otherwise, turn light off.
- stage 2, from day 101: as before, except that:
  counter twice interrogated on day n counts until $100 - n$ only;
  non-counters who only saw light off in stage 1: do nothing;
  non-counters who saw light on in stage 1: do the usual. (24 y)

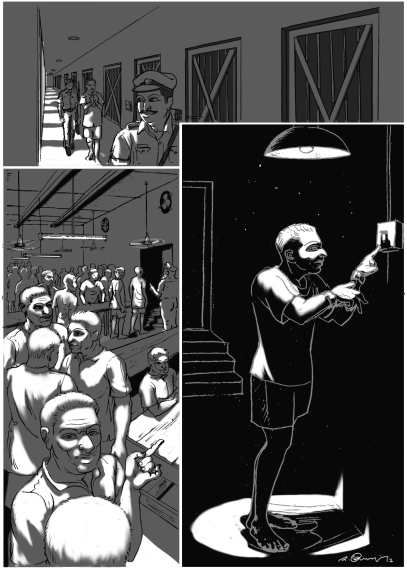
Head counter and assistant counters (iterated protocol, 2 stages):

- stage 1: head and assistant counters count to agreed max. n;
- stage 2: head counter collects from *successful* assistants;
- repeat stage 1 (unsuccessful assistants continue counting to n) and stage 2 (not yet collected successful assistants, and newly successful assistants) until termination. (9 years)

Minimum not known!

One hundred prisoners ...





More information

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