Consecutive numbers

Anne and Bill are each going to be told a natural number. Their numbers will be one apart. The numbers are now being whispered in their respective ears. They are aware of this scenario. Suppose Anne is told 2 and Bill is told 3. The following truthful conversation between Anne and Bill now takes place:

- Anne: "I do not know your number."
- Bill: "I do not know your number."
- Anne: "I know your number."
- Bill: "I know your number."

Explain why is this possible.

Consecutive numbers — representing uncertainties

$\underline{(2,3)}$

Consecutive numbers — representing uncertainties

$$(2,1) - a - \underline{(2,3)}$$

Consecutive numbers — representing uncertainties

$$(2,1) - a - \underline{(2,3)} - b - (4,3)$$

Consecutive numbers — representing uncertainties

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3)$$

Consecutive numbers — representing uncertainties

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

Consecutive numbers — representing uncertainties

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \longrightarrow \cdots$$

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \longrightarrow \cdots$$

Consecutive numbers — representing uncertainties

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne knows that her number is 2.
- Bill knows that Anne's number is 2 or 4.
- Anne and Bill commonly know that Bill's number is odd.
- ...

Consecutive numbers — successive announcements

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: "I do not know your number." ??

Consecutive numbers — successive announcements

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(0,1) - b - (2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: “I do not know your number.” eliminated states

Consecutive numbers — successive announcements

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: "I do not know your number."

Consecutive numbers — successive announcements

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: "I do not know your number."
- Bill: "I do not know your number." ??

Consecutive numbers — successive announcements

$$(1,0) - a - (1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$(2,1) - a - \underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: "I do not know your number."
- Bill: "I do not know your number." eliminated states

Consecutive numbers — successive announcements

$$(1,2) - b - (3,2) - a - (3,4) \text{ ——————— } \cdots$$

$$\underline{(2,3)} - b - (4,3) \text{ ——————— } \cdots$$

- Anne: “I do not know your number.”
- Bill: “I do not know your number.”

Consecutive numbers — successive announcements

$$(1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$\underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: "I do not know your number."
- Bill: "I do not know your number."
- Anne: "I know your number." ??

Consecutive numbers — successive announcements

$$(1,2) - b - (3,2) - a - (3,4) \text{———} \cdots$$

$$\underline{(2,3)} - b - (4,3) \text{———} \cdots$$

- Anne: "I do not know your number."
- Bill: "I do not know your number."
- Anne: "I know your number." eliminated states

Consecutive numbers — successive announcements

(1,2)

<u>(2,3)</u>

- Anne: "I do not know your number."
- Bill: "I do not know your number."
- Anne: "I know your number."

Consecutive numbers — successive announcements

(1,2)

$\underline{(2,3)}$

- Anne: “I do not know your number.”
- Bill: “I do not know your number.”
- Anne: “I know your number.”
- Bill: “I know your number.” ??

Consecutive numbers — successive announcements

(1,2)

(2,3)

- Anne: “I do not know your number.”
- Bill: “I do not know your number.”
- Anne: “I know your number.”
- Bill: “I know your number.” already common knowledge

Consecutive numbers — successive announcements

(1,2)

$\underline{(2,3)}$

- Anne: "I do not know your number."
- Bill: "I do not know your number."
- Anne: "I know your number."
- Bill: "I know your number."

An alternative representation

Anne and Bill are each going to be told a natural number. Their numbers will be one apart. The numbers are now being whispered in their respective ears. They are aware of this scenario. Suppose Anne is told 2 and Bill is told 3.

Anne and Bill each have a natural number on their forehead. Their numbers are one apart. They only can see the number on the other's forehead. They are aware of this scenario. Suppose Anne has the number 3 and Bill has the number 2.

Intermezzo — What is my number?

Each of agents Anne, Bill, and Cath has a positive integer on its forehead. They can only see the foreheads of others. One of the numbers is the sum of the other two. All the previous is common knowledge. The agents now successively make the truthful announcements:

1. *Anne: "I do not know my number."*
2. *Bill: "I do not know my number."*
3. *Cath: "I do not know my number."*
4. *Anne: "I know my number. It is 50."*

What are the other numbers?

Math Horizons, november 2004, probleem 182.

What is my number?

When does Anne know her number, initially?

What is my number?

When does Anne know her number, initially?
When the numbers had been $(2, 1, 1)$,
Anne sees the numbers 1 and 1;

What is my number?

When does Anne know her number, initially?

When the numbers had been $(2, 1, 1)$,
Anne sees the numbers 1 and 1;
her number must be 2 or 0;
0 is excluded.
So she knows it must be 2!

What is my number? — a look at the model

- domain of triples (x, y, z) such that $x = y + z$ or $y = x + z$ or $z = x + y$
- Anne cannot distinguish $(y + z, y, z)$ from $(|y - z|, y, z)$
- Bill cannot distinguish ...
- Cath cannot distinguish ...

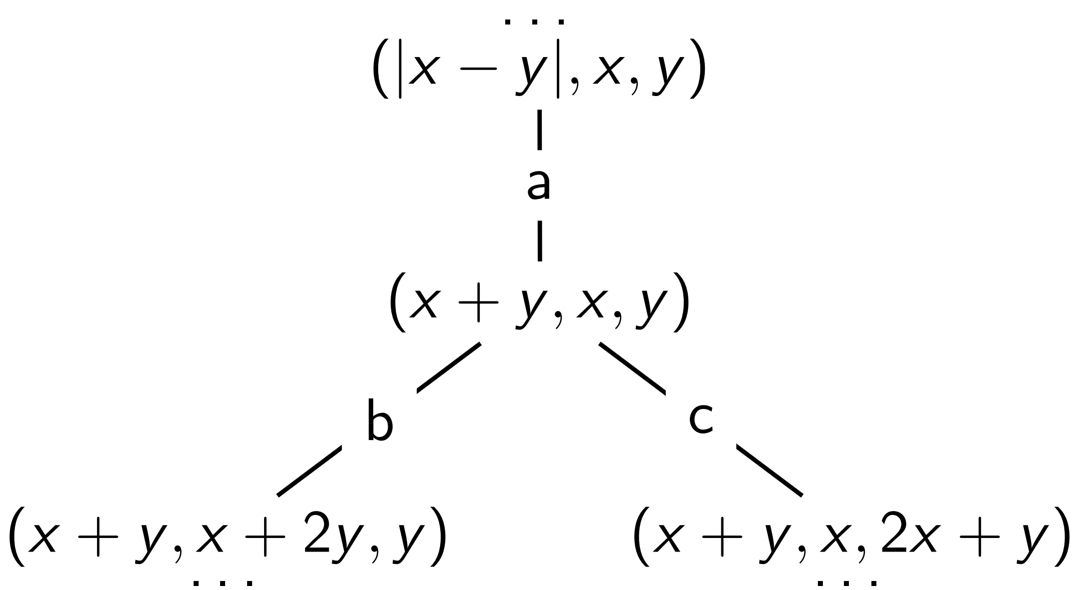
Example:

$$\begin{tikzcd}
 & \cdots & \\
 & (65, 143, 78) \arrow[d, dash, "a"] & \\
 & (221, 143, 78) \arrow[dl, dash, "b"'] \arrow[dr, dash, "c"] & \\
(221, 299, 78) & & (221, 143, 364) \\
\cdots & & \cdots
\end{tikzcd}$$

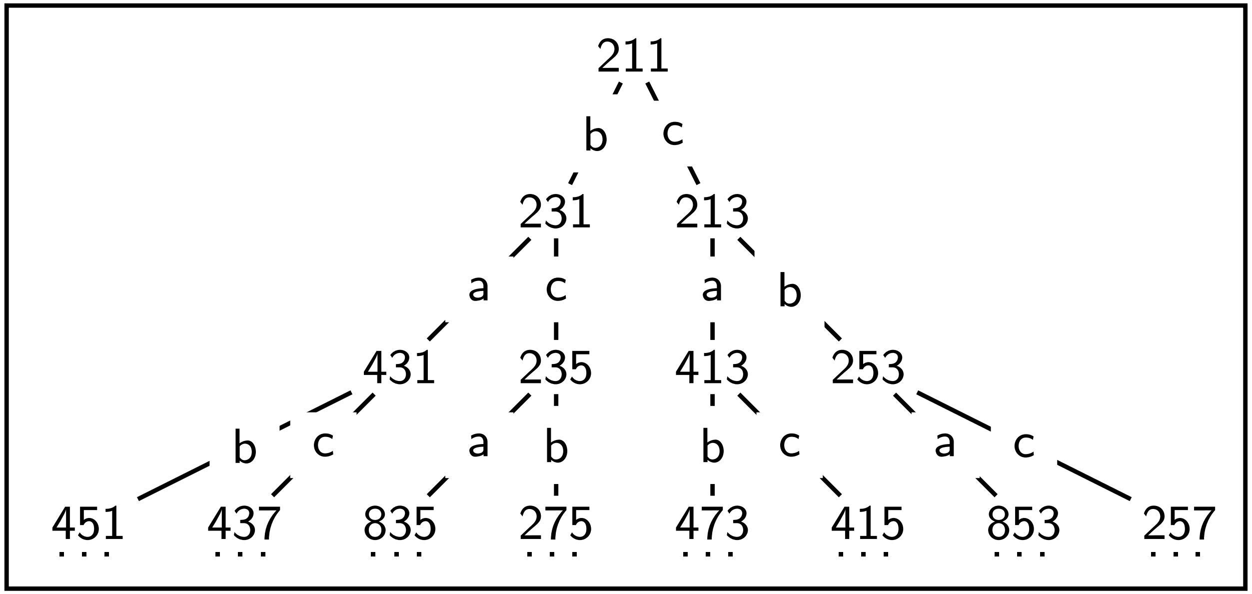
What is my number? — a look at the model

- domain of triples (x, y, z) such that $x = y + z$ or $y = x + z$ or $z = x + y$
- Anne cannot distinguish $(y + z, y, z)$ from $(|y - z|, y, z)$
- Bill cannot distinguish ...
- Cath cannot distinguish ...

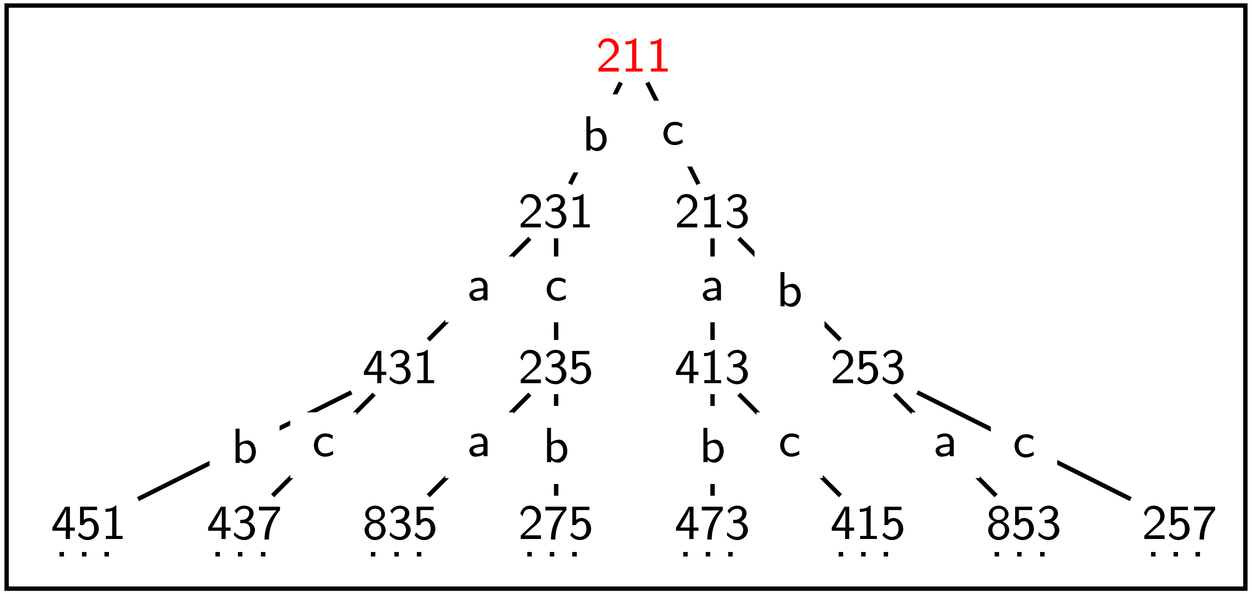
Example:



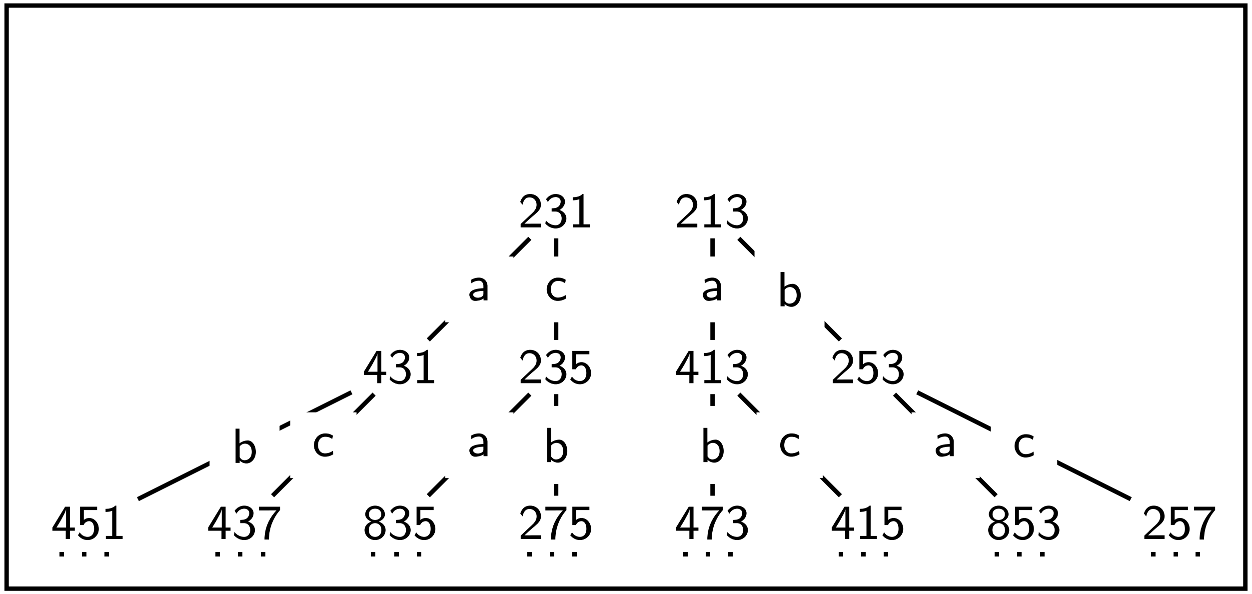
Revealing ignorance



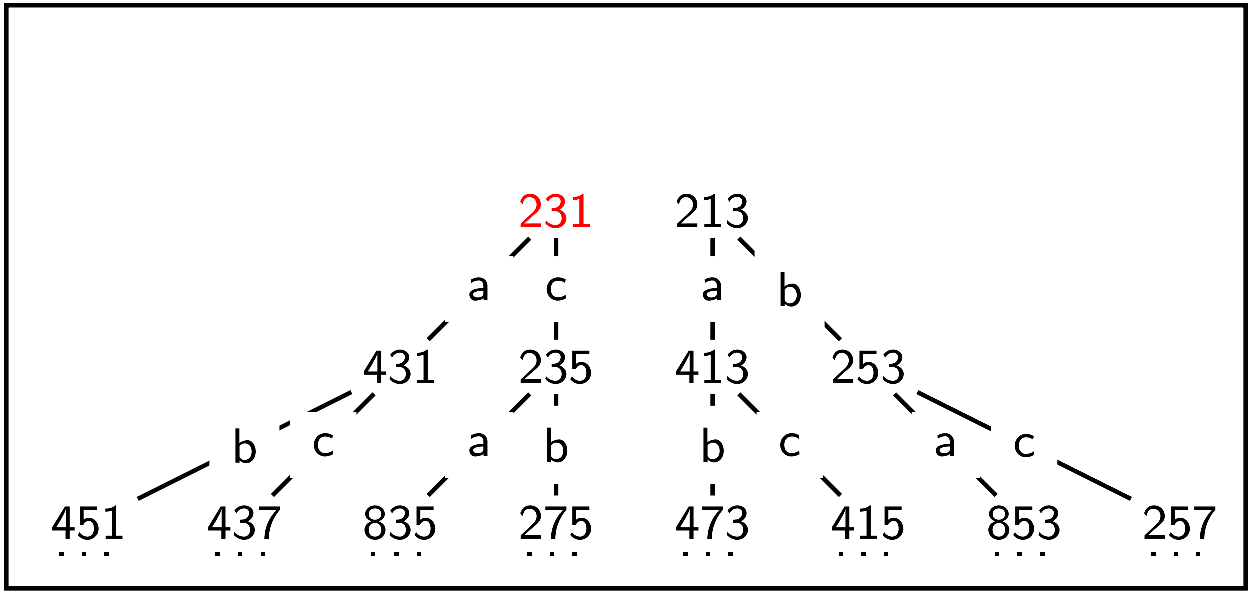
Revealing ignorance



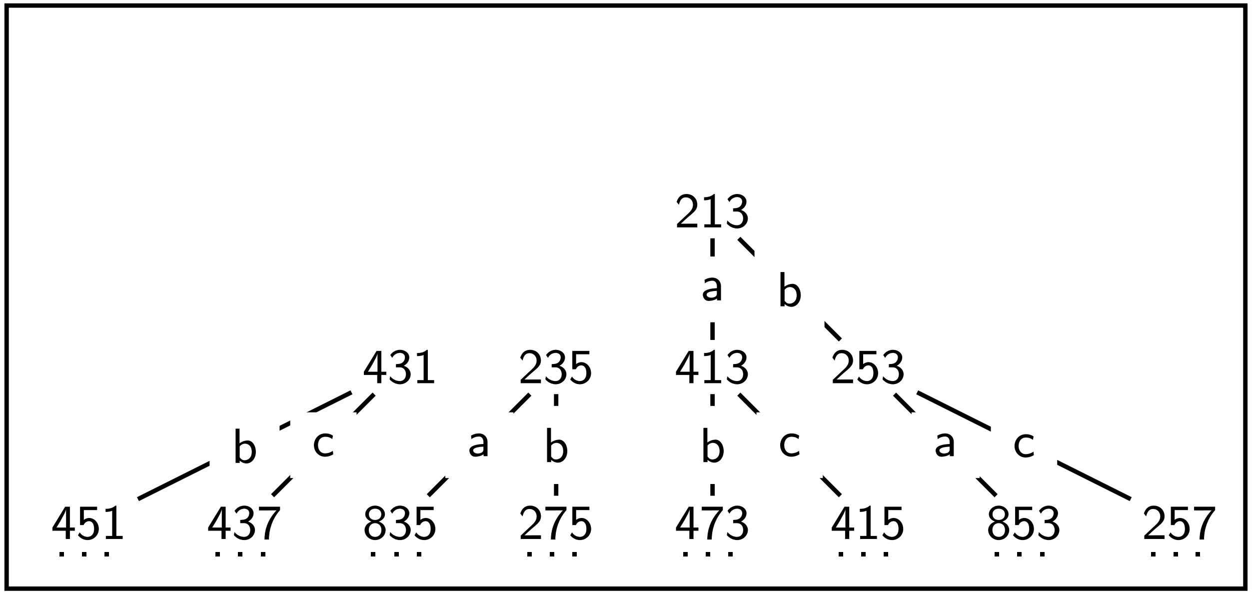
Revealing ignorance



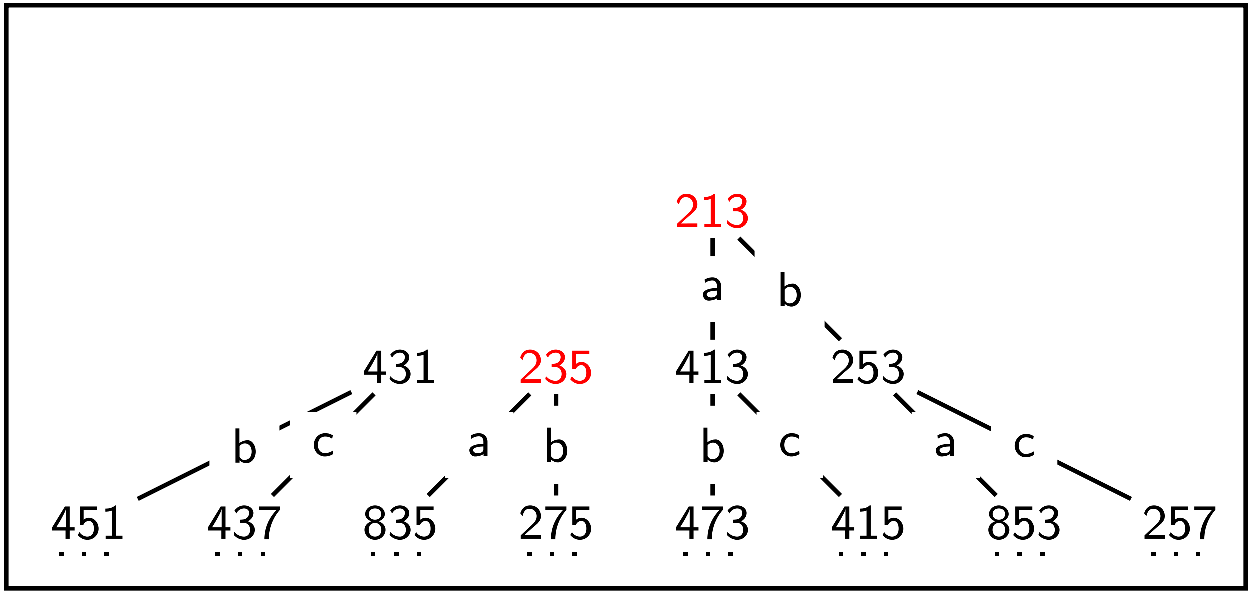
Revealing ignorance



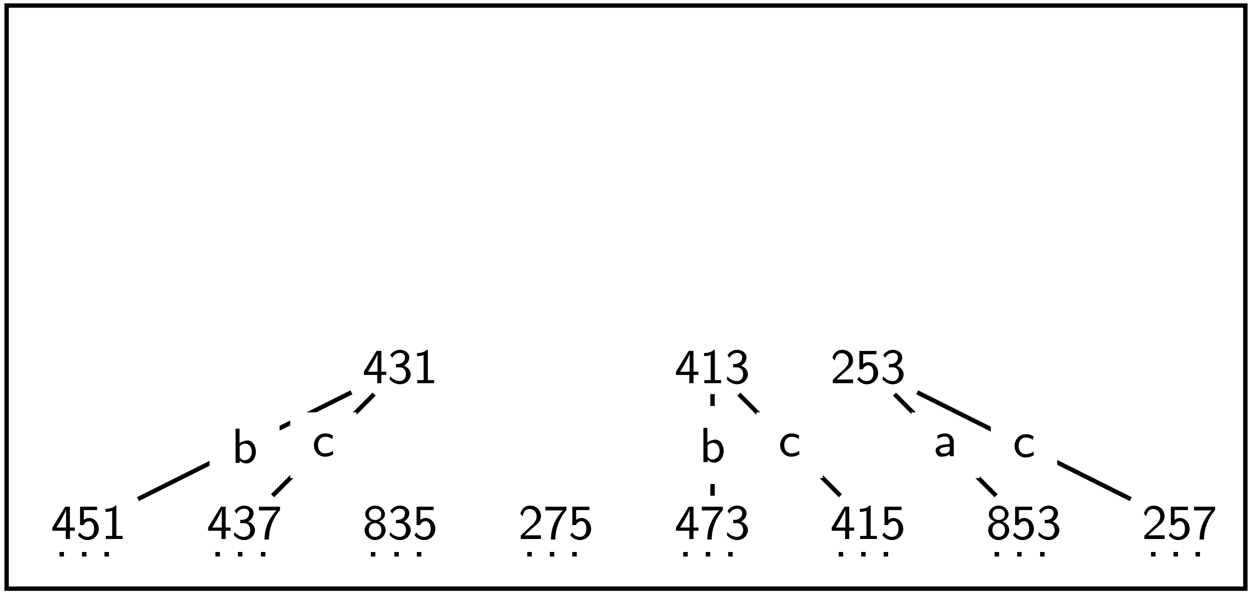
Revealing ignorance



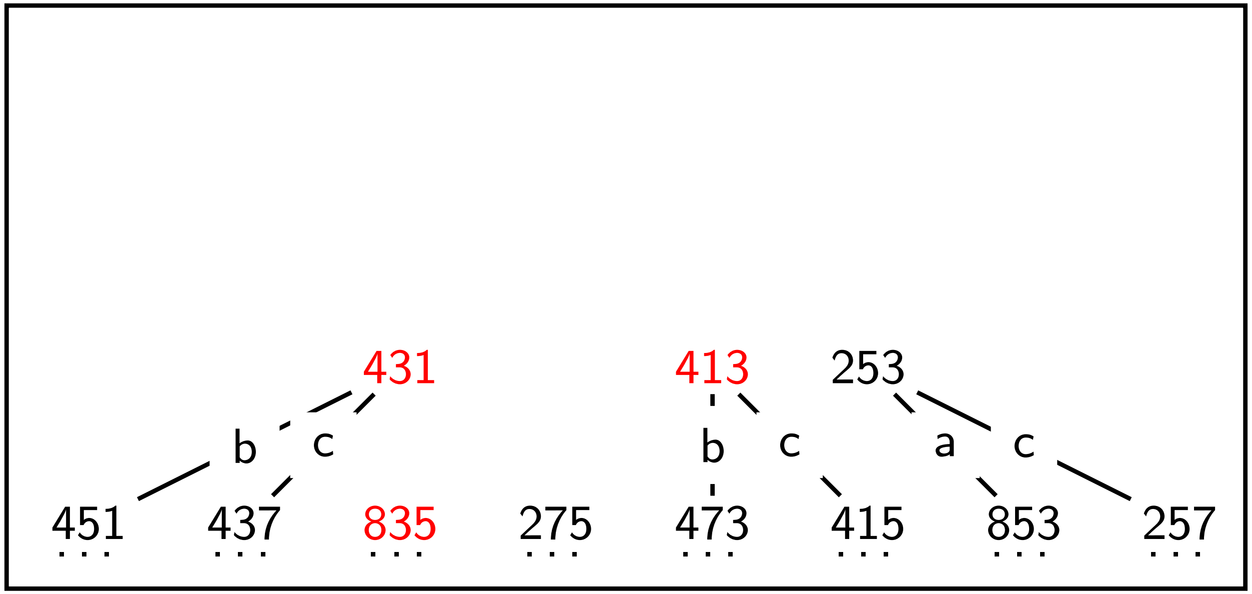
Revealing ignorance



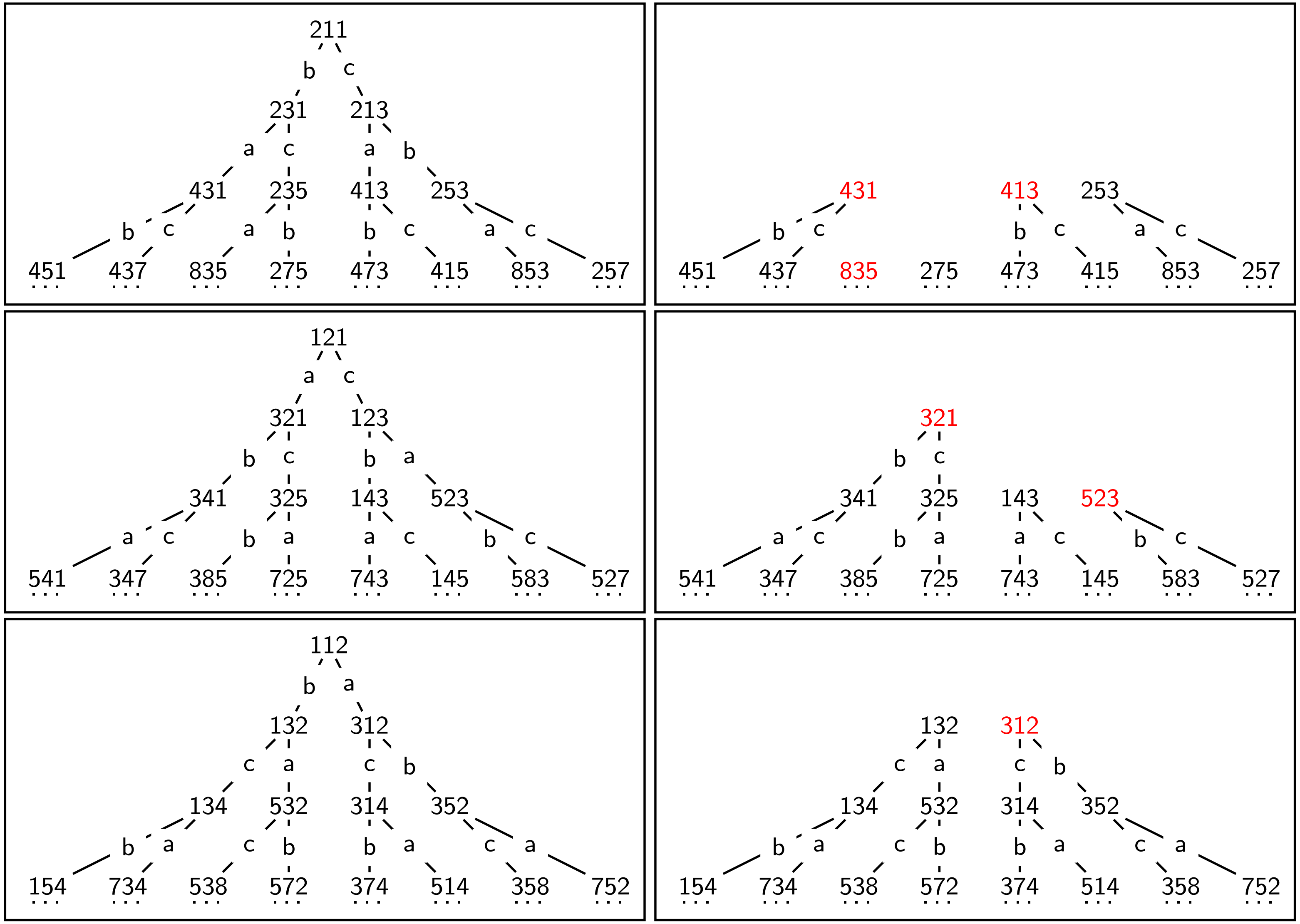
Revealing ignorance



Revealing ignorance



Where after three ignorance announcements Anne knows



What is my number? — different wording of the riddle

1. Anne: “I do not know my number.”
2. Anne: “I do not know my number.”
3. Anne: “I do not know my number.”
4. Anne: “I know my number. It is 50.”

What are the other numbers?

What is my number? — different wording of the riddle

1. Anne: “I do not know my number.”
2. Bill: “I do not know my number.”
3. Cath: “I do not know my number.”
4. ~~Anne: “I know my number. It is 50.~~”

~~What are the other numbers?~~
What are the numbers, if Anne now knows her number and the numbers are all prime?

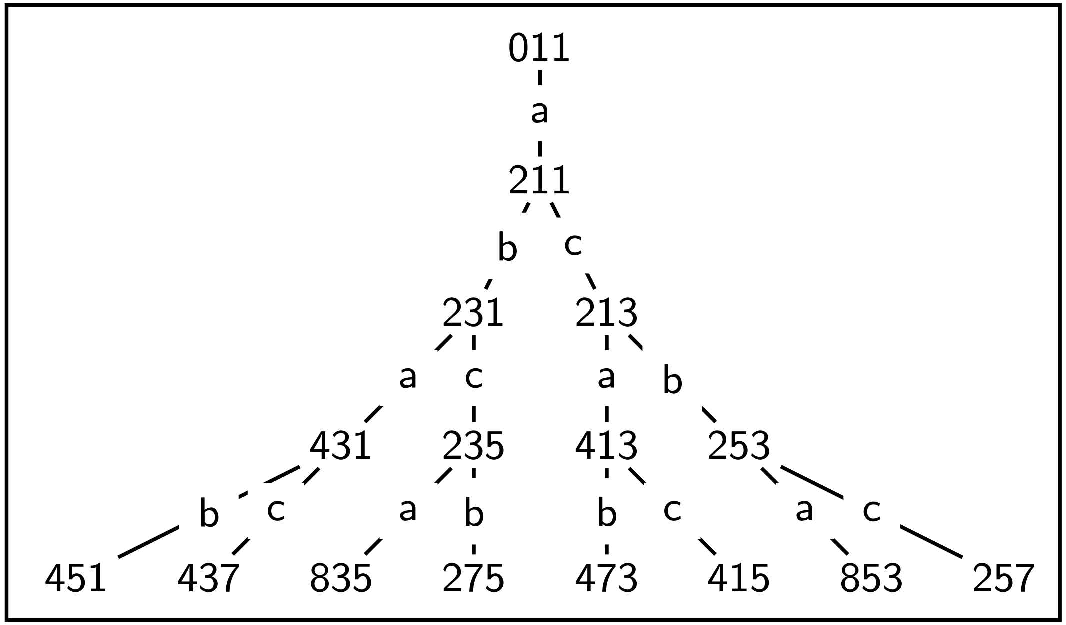
What is my number? — when 0 is also allowed

Each of agents Anne, Bill, and Cath has a ~~positive integer~~ **natural number** on its forehead. They can only see the foreheads of others. One of the numbers is the sum of the other two. All the previous is common knowledge. The agents now successively make the truthful announcements:

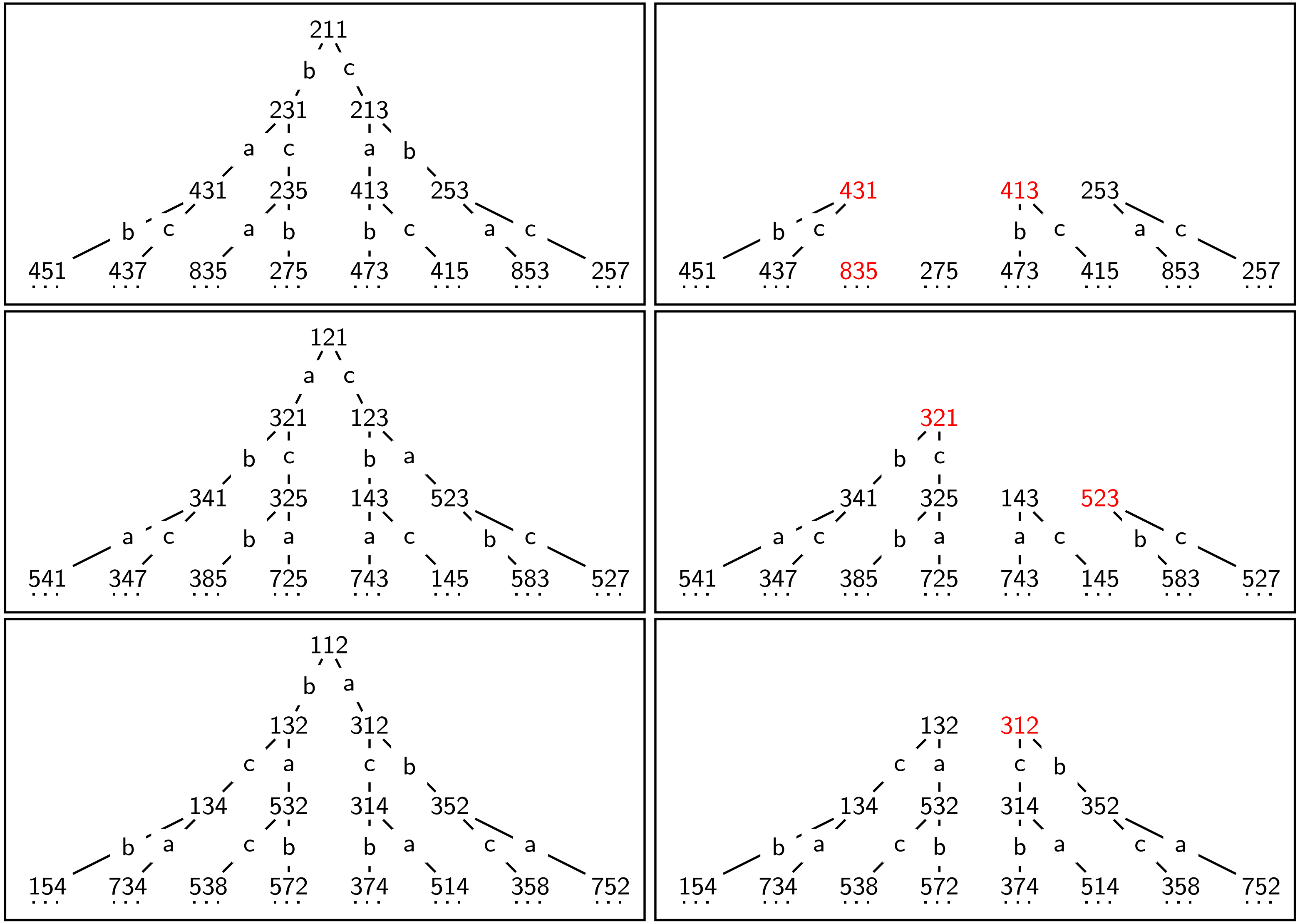
1. Anne: "I do not know my number."
2. Anne: "I do not know my number."
3. Anne: "I do not know my number."
4. Anne: "I know my number. It is 50."

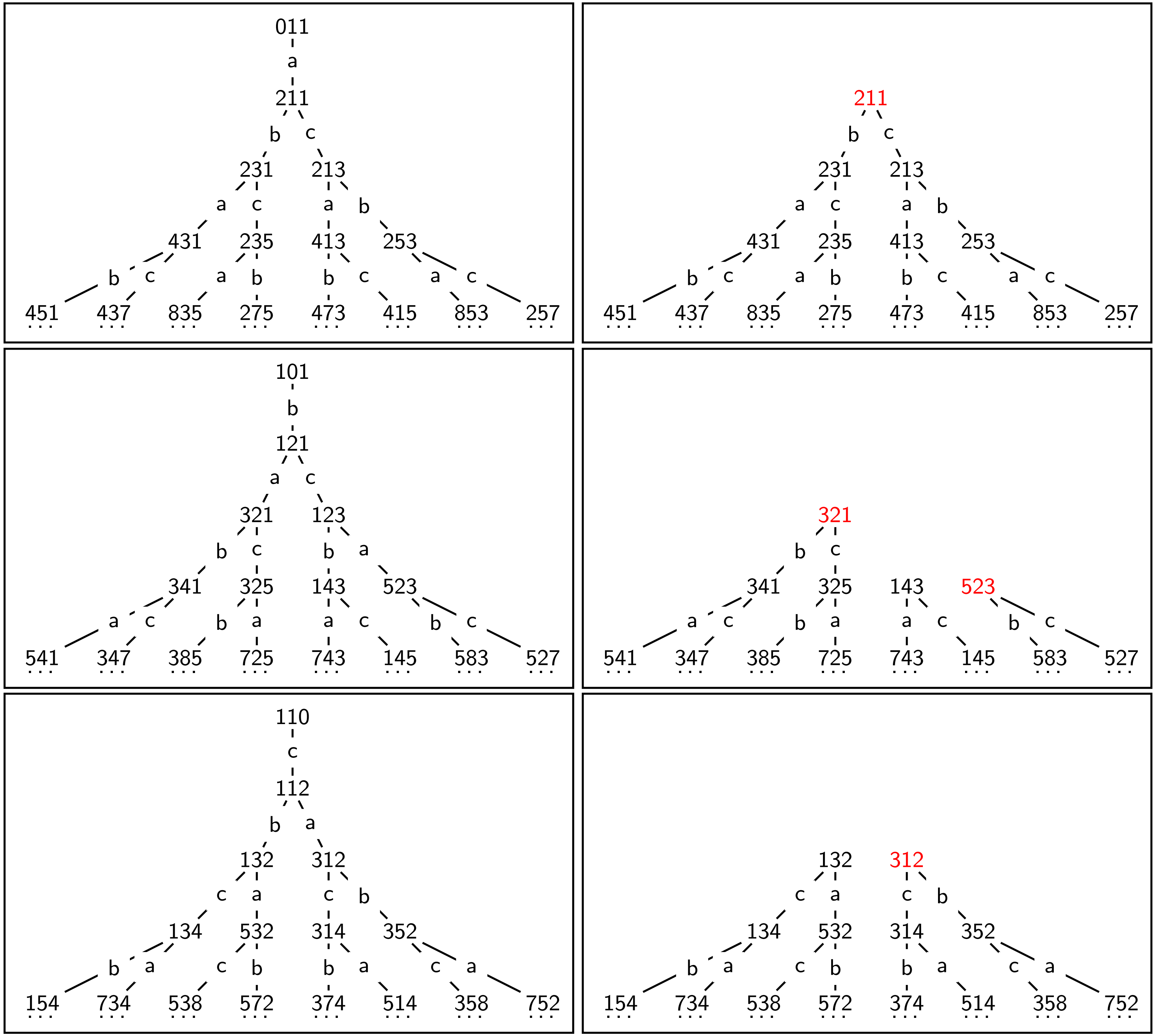
What are the other numbers?
This can no longer be determined? Why?

What is my number? — when 0 is also allowed



Where after three ignorance announcements Anne knows




011
a
211
b c
231 213
a c a b
431 235 413 253
b c a b b c a c
451 437 835 275 473 415 853 257
211
b c
231 213
a c a b
431 235 413 253
b c a b b c a c
451 437 835 275 473 415 853 257
101
b
121
a c
321 123
b c b a
341 325 143 523
a c b a a c b c
541 347 385 725 743 145 583 527
321
b c
341 325 143 523
a c b a a c b c
541 347 385 725 743 145 583 527
110
c
112
b a
132 312
c a c b
134 532 314 352
b a c b b a c a
154 734 538 572 374 514 358 752
132 312
c a c b
134 532 314 352
b a c b b a c a
154 734 538 572 374 514 358 752

Sum and product

A says to S and P: I have chosen two integers x, y such that $1 < x < y$ and $x + y \leq 100$. In a moment, I will inform S only of $s = x + y$, and P only of $p = xy$. These announcements remain private. You are required to determine the pair (x, y).
He acts as said. The following conversation now takes place:

1. P says: "I do not know it."
2. S says: "I knew you didn't."
3. P says: "I now know it."
4. S says: "I now also know it."

Determine the pair (x, y).

Sum and product — history

Originally stated, in Dutch, by Hans Freudenthal.
Nieuw Archief voor Wiskunde 3(17):152, 1969.
Became popular in AI by way of John McCarthy, Martin Gardner.

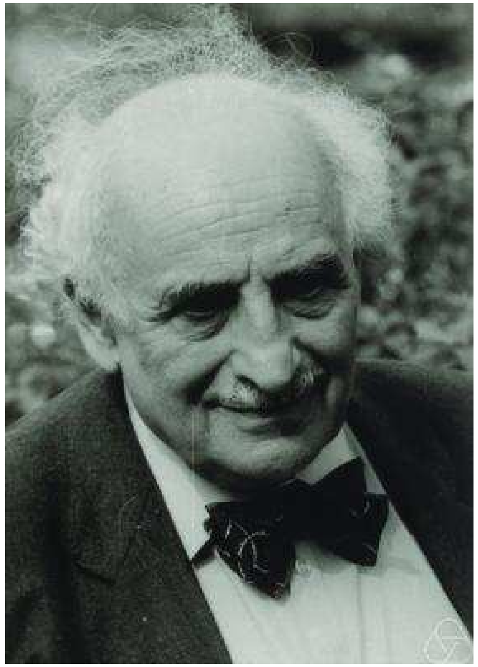
No. 223. A zegt tot S en P: Ik heb twee gehele getallen x, y gekozen met $1 < x < y$ en $x + y \leqslant 100$. Straks deel ik $s = x + y$ aan S alleen mee, en $p = xy$ aan P alleen. Deze mededelingen blijven geheim. Maar jullie moeten je inspannen om het paar (x, y) uit te rekenen.

Hij doet zoals aangekondigd. Nu volgt dit gesprek:

1. P zegt: Ik weet het niet.
2. S zegt: Dat wist ik al.
3. P zegt: Nu weet ik het.
4. S zegt: Nu weet ik het ook.

Bepaal het paar (x, y).

(*H. Freudenthal*).



Towards a solution: first announcement

A says to S and P: I have chosen two integers x, y such that $1 < x < y$ and $x + y \leq 100$. In a moment, I will inform S only of $s = x + y$, and P only of $p = xy$. These announcements remain private. You are required to determine the pair (x, y).
He acts as said. The following conversation now takes place:

1. ~~P says: "I do not know it."~~
2. S says: "I knew you didn't."
3. P says: "I now know it."
4. S says: "I now also know it."

Determine the pair (x, y).

All four announcements are informative.
The second announcement implies the first announcement.

Towards a solution: $(2, 3)$ and $(14, 16)$

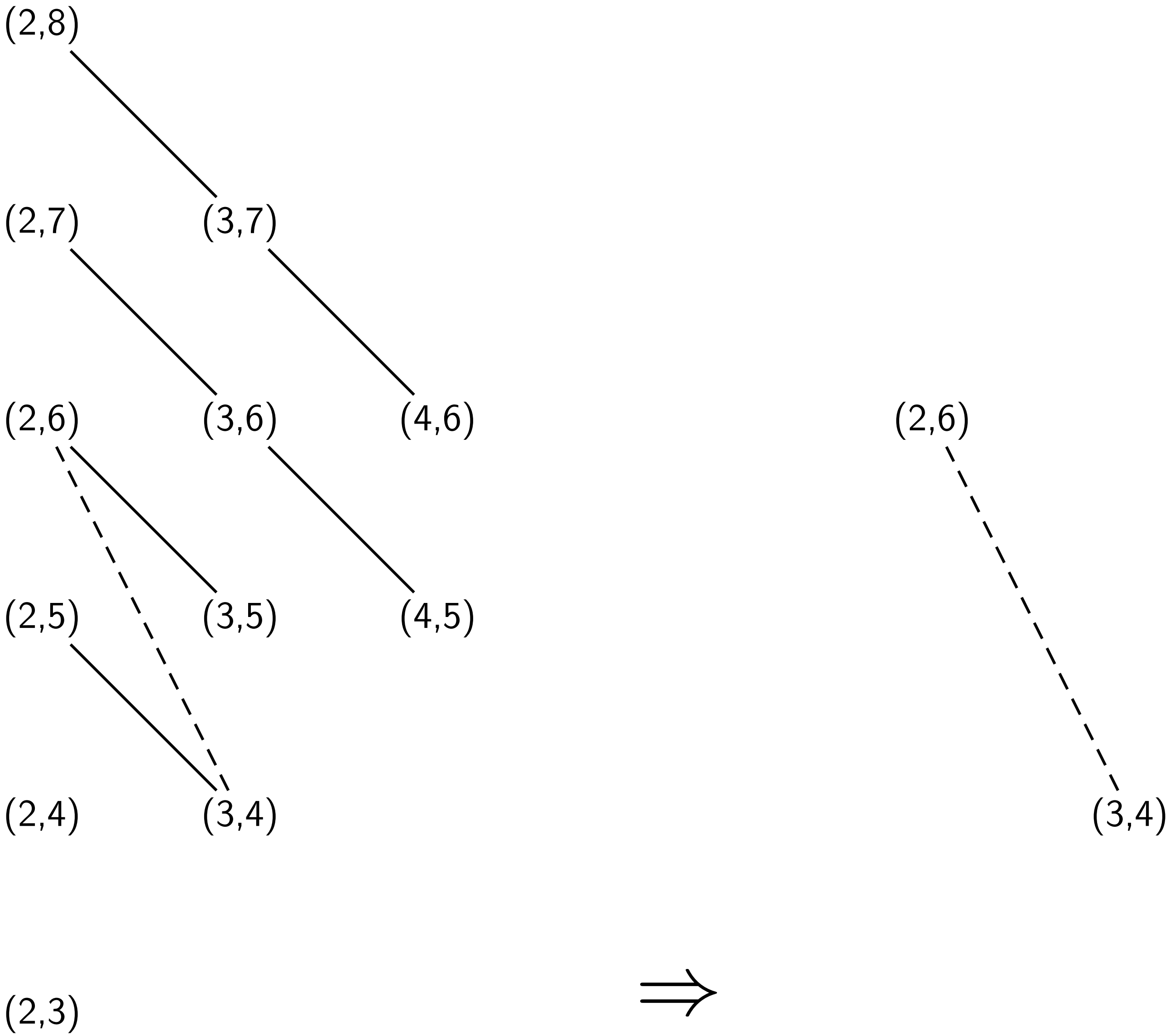
If the numbers were 2 and 3, then P deduces the pair from their product: $6 = 2 \cdot 3$ and $6 = 1 \cdot 6$, but the numbers are larger than 1 (two integers x, y such that $1 < x < y$ and $x + y \leq 100$).

If the numbers were prime, then P deduces the pair, because of the unique factorization of the product.

If the numbers were 14 and 16, then their sum 30 is also the sum of 7 and 23. If they had been 7 and 23 P would know the numbers. But if they had been 14 and 16, P would not know the numbers because this is also the product of 7 and 32, or of 28 and 8 (and also of 2 and 102: but that's out, because $2 + 102 > 100$!). Therefore, S considers is possible that P knows the numbers and that P does not know the numbers. In other words: S does not know that P does not know the numbers.

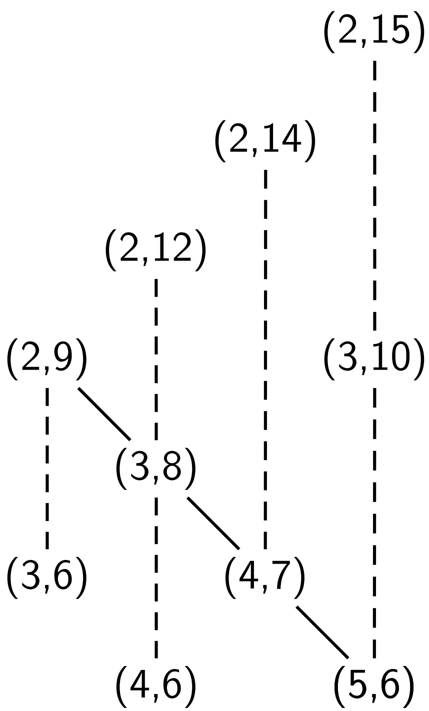
Towards a solution: $1 < x < y$ and $x + y \leq 10$

P: "I do not know it." S: "I now know it." P: "I do not know it."



For sum 11 S knows that P does not know

sum 11	*product*	*other sum, same product*
$(2, 9)$	18	$(3, 6)$
$(3, 8)$	24	$(4, 6), (2, 12)$
$(4, 7)$	28	$(2, 14)$
$(5, 6)$	30	$(2, 15), (3, 10)$



Second announcement: S says: "I knew you didn't."

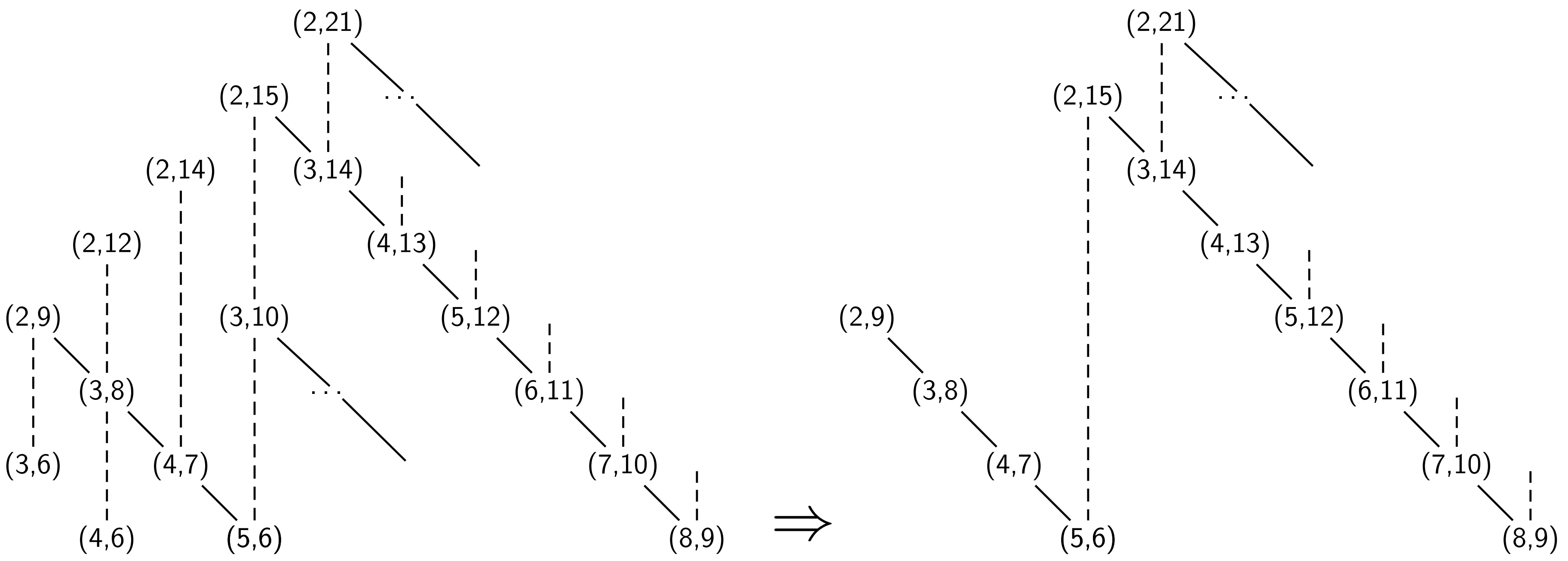
Remaining sums are

$$11, 17, 23, 27, 29, 35, 37, 41, 47, 53$$

For sum 11, see previous slide. For another example, sum 17:

sum 17	*product*	*other sum, same product*
$(2, 15)$	30	$(3, 10), (5, 6)$
$(3, 14)$	42	$(2, 21), (6, 7)$
$(4, 13)$	52	$(2, 26)$
$(5, 12)$	60	$(2, 30), (3, 20), (4, 15), (6, 10)$
$(6, 11)$	66	$(2, 33), (3, 22)$
$(7, 10)$	70	$(2, 35), (5, 14)$
$(8, 9)$	72	$(2, 36), (3, 24), (4, 18)$

Second announcement: S says: "I knew you didn't."



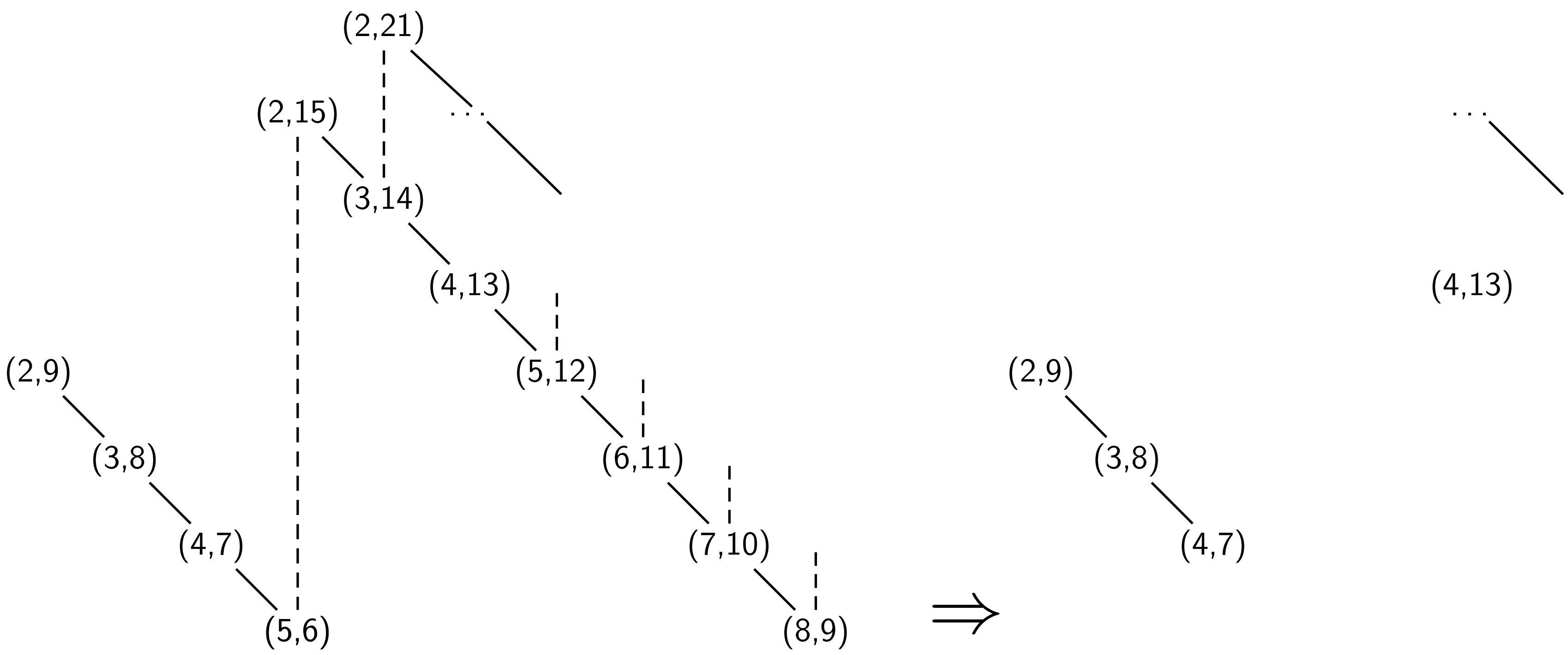
Third announcement: P says: "I now know it."

Call a number pair **closed** if P knows what the numbers are. Otherwise, it is **open**. Because P says he now knows, all open pairs can be eliminated. Example for sum 17:

sum 17	*product*	*other sum, same product*
$(2, 15)$	30	$(5, 6)$
$(3, 14)$	42	$(2, 21)$
$(4, 13)$	52	–
$(5, 12)$	60	$(3, 20)$
$(6, 11)$	66	$(2, 33)$
$(7, 10)$	70	$(2, 35)$
$(8, 9)$	72	$(3, 24)$

Third announcement: *P* says: “I now know it.”

Remove open pairs.



Fourth announcement: S says: “I now also know it.”

Of the remaining lines with the same sum, the one for sum 17 is the only one that contains a single pair, $(4, 13)$.