What does a group know?

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joint works with
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Background: Group Knowledge

We assume given

a finite set $N = \{1, \ldots, n\}$ of agents

a countably infinite set of primitive propositions

Epistemic/doxastic logic

A model is a tuple $M = \langle W, \sim_1, \ldots, \sim_n, V \rangle$:

- W is a set of states
- $\sim_i$ is an epistemic accessibility relation
  - Sometimes assumed to be an equivalence relation (S5)
  - Sometimes assumed to be transitive, euclidian and serial (KD45)
- V is a valuation function, assigning primitive propositions to each state

Epistemic logic

Language: $\phi ::= p \mid K_i\phi \mid \neg\phi \mid \phi_1 \wedge \phi_2$

Interpretation: $(M, s) \models K_i\phi$ iff for all t s.t. $s \sim_i t$, $(M, t) \models \phi$

What does a group know?

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$$(M, s) \models D_G\phi \text{ iff } s \sim_G^D t \Rightarrow (M, t) \models \phi, \text{ where } \sim_G^D = \bigcap_{i \in G} \sim_i$$

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$$(M,s) \models C_G\phi \text{ iff } s \sim_G^C t \Rightarrow (M,t) \models \phi, \text{ where } \sim_G^C = (\bigcup_{i\in G} \sim_i)^*$$

Distributed Knowledge: Key Axioms

$$D_A\phi \to D_B\phi \text{ when } A \subseteq B$$

$$D_{\{a\}}\phi \leftrightarrow K_a\phi$$

Key research issue

The relationship between these notions of group knowledge is not well understood

Generalised Distributed Knowledge
(based on joint work with Dmitry Shkatov)

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Distributed knowledge

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- In other words, the group considers a state
  - impossible iff at least one member of the group considers it impossible
  - possible iff all the agents in the group considers it possible
- For S5 agents this makes sense
  - If an S5 agent considers a state impossible, then it is impossible
    - .. and this is common knowledge

Distributed knowledge for non-S5 agents

$$\sim_G^D = \bigcap_{i \in G} \sim_i$$

- The group considers a state
  - impossible iff at least one member of the group considers it impossible
  - possible iff all the agents in the group considers it possible
- For non-S5 agents, in particular agents without T/reflexivity (e.g., KD45):
  - If one agent considers a state impossible, that agent might in fact be wrong
  - Ruling out a state based on the evidence of a single agent is then a very credulous group attitude
  - Curious asymmetry between the evidence need for possibility vs. impossibility
    - impossibility: every agent is a veto voter, possibility: unanimity

Generalised distributed knowledge

- In this work we look at general definitions of distributed knowledge where we vary the evidence needed for the two cases

Generalised Distributed Knowledge

- The group considers a state
  - impossible iff not at least k agents in the group considers it impossible
  - possible iff at least k agents in the group considers it possible

The generalised distributed knowledge operator

$$M, s \models D_G^{+k}\phi \Leftrightarrow \forall (s,t) \in \sim_G^{+k} M, t \models \phi$$

$$\sim_G^{+k} = \bigcup_{H \subseteq G, |H| \geq k} \bigcap_{i \in H} \sim_i$$

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$$\text{E.g., } \sim_G^{maj} = \sim_G^{+\lceil (|G|+1)/2 \rceil}$$

Expressive power and succinctness

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid D_G\phi \mid D_G^{+k}\phi$$

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$$(M,s) \models D_G^{+k}\phi \Leftrightarrow (M,s) \models \bigwedge_{H\subseteq G, |H|\geq k} D_H\phi$$

Generalised distributed knowledge

- Not more expressive than standard distributed knowledge
- But exponentially more succinct

Generalised distributed knowledge: the extremes

$$\sim_G^{+k} = \bigcup_{H \subseteq G, |H| \geq k} \bigcap_{i \in H} \sim_i$$

Generalised distributed knowledge: the extremes

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  - impossible iff at least one member of the group considers it impossible
  - possible iff all the agents in the group considers it possible
- k = 1: the group considers a state
  - impossible iff all agents in the group considers it impossible
  - possible at least one agent in the group considers it possible

Generalised distributed knowledge: the extremes

$$\sim_G^{+k} = \bigcup_{H \subseteq G, |H| \geq k} \bigcap_{i \in H} \sim_i$$

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  - impossible iff at least one member of the group considers it impossible
  - possible iff[standard distributed knowledge]ble
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Generalised distributed knowledge: the extremes

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- k = |G|: the group considers a state $\sim_G^{+|G|} = \sim_G^D$
  - impossible iff at least one member of the group considers it impossible
  - possible iff standard distributed knowledge ble
- k = 1: the group considers a state $\sim_G^{+1} = \sim_G^E$
  - impossible iff all agents in the group considers it impossible
  - possib general knowledge (everybody knows) e

Complexity

- Theorem: the satisfiability problem is PSPACE-complete as long as the underying logic is PSPACE-complete (true for any logic between K and S4, as well as KD45).

Generalised distributed knowledge: conclusions

- Between distributed and general knowledge
    - Intuitively two entirely different concepts
    - But we show that the difference between them can be explained quantitatively rather than qualitatively
    - Specific instances of the same concept, corresponding to which voting threshold is used
    - There is a scale of intermediate concepts between them